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On the number operators of multimode systems of deformed oscillators covariant under quantum groups

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Abstract. For multimode systems of deformed oscillators covariant under the actions of the quantum groups $\mathrm{SU}_q(n)$, $\mathrm{SU}_q(n|m)$, $\mathrm{GL}_{p,q}(n)$ and $\mathrm{GL}_{p,q}(n|m)$ the number operators are constructed explicitly in terms of the creation and annihilation operators. The relation between the various kinds of deformed oscillator systems, representations of these oscillator algebras in terms of coordinates and deformed derivatives, realizations of classical groups in non-commutative spaces, and some aspects of the physical behaviour of quantum group covariant oscillator systems are also discussed.

1. Introduction

The theory of quantum groups [1–7] has led to the generalizations (deformations) of the oscillator (boson, fermion) algebras in several directions. The development of differential calculus in non-commutative (quantized) spaces has identified multimode/multidimensional systems of deformed (bosonic, fermionic) creation/ coordinate and annihilation/ derivative operators covariant under the actions of quantum groups [5–7]. Generalization of the usual boson–fermion realizations to quantized Lie algebras and superalgebras have resulted in the study of single-mode deformed bosons [8,9] and fermions [9–11]. The relation between the various types of deformed oscillators has also been clarified [12–14] as somewhat similar to the well-known statistics-altering Jordan–Wigner/Klein transformations.

A single-mode q-oscillator with the creation ($\alpha^\dagger$), annihilation (α) and number (N) operators obeying the relations

$$[\mathrm{N}, \alpha^\dagger] = \alpha^\dagger \qquad [\mathrm{N}, \alpha] = -\alpha \qquad \alpha\alpha^\dagger - q\alpha^\dagger\alpha = 1 \tag{1.1}$$

where the deformation parameter $q \in [-1, \infty)$ is real, has also been the subject of study by some authors [15–16] in the past also, independent of the recent

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developments due to the theory of quantum groups. A multimode generalization of the q-oscillator, with $-1 \leqslant q \leqslant 1$

$$\alpha_i \alpha_j^\dagger - q \alpha_j^\dagger \alpha_i = \delta_{ij} \qquad i, j = 1, 2, \ldots \tag{1.2}$$

and no commutation relations imposed between the (α_i, α_j)s, plays a central role in modelling a possible small violation of the usual Bose/Fermi statistics [17, 18]; note that the algebra (1.2) is intermediate between the bosonic ($q = 1$) and the fermionic ($q = -1$) cases. The relations (1.2) are easily seen to be covariant under the linear transformations of the classical special unitary group (i.e. $\alpha_i \to \sum_j U_{ij} \alpha_j$, $[U_{ij}] \in$ SU group). A set of coupled q-oscillators obeying relations similar to (1.2) has also been considered earlier [15] in the context of particle physics phenomenology. A two-parameter generalization of the system (1.2) has also been studied recently [19] from the point of view of exploring possible new forms of quantum statistics.

In contrast to a system of the type (1.2), the quantum group covariant systems, with which we shall be concerned here, are not covariant under the actions of the classical groups and have the coupling between the various modes different from (1.2) (see (2.5) below). In this paper we address the problem of constructing the number operators for individual modes of multimode systems of oscillators covariant under the actions of certain quantum groups, explicitly, in terms of the corresponding creation and annihilation operators. The key theorem helping us solve this problem is that for the q-oscillator (1.1) the number operator can be written [20] as

$$\mathrm{N} = \sum_{k=1}^{\infty} \frac{(1-q)^k}{(1-q^k)} (\alpha^\dagger)^k \alpha^k \tag{1.3}$$

for any $q \in [-1, \infty)$; for $q = -1$, i.e. for a fermion, one should remember the condition $\alpha^2 = (\alpha^\dagger)^2 = 0$.

This paper is organized as follows. In section 2, after a brief review of the definition and the Fock representation of $\mathrm{SU}_q(n)$-covariant systems of oscillators, we present the expressions for the corresponding number operators. In section 3 we briefly review the relation between the $\mathrm{SU}_q(n)$-covariant system and the various sets of independent q-oscillators using the formalism of section 2. Section 4 describes a procedure for constructing the coherent states of the individual modes for the $\mathrm{SU}_q(n)$-covariant multimode system. In section 5 the known realization of the single-mode q-oscillator (1.1) in terms of the Jackson q-derivative operator is generalized leading to a similar realization of the $\mathrm{SU}_q(n)$-covariant system of oscillators. In sections 6 and 7 the multimode systems covariant under the quantum group $\mathrm{GL}_{p,q}(n)$, and the quantum supergroups $\mathrm{SU}_q(n|m)$ and $\mathrm{GL}_{p,q}(n|m)$, are studied along the same lines as the study of $\mathrm{SU}_q(n)$-covariant systems in sections 2–5. In section 8 it is shown how classical groups can be realized in non-commutative spaces using q-oscillator algebras. To conclude, in section 9 some interesting observations are made on the physical behaviour of the multimode systems of oscillators with quantum group covariance.

2. The $\mathrm{SU}_q(n)$-covariant multimode system of q-oscillators

An $\mathrm{SU}_q(2)$-matrix can be written in the form

$$\begin{pmatrix} a & qb \\ -b^\star & a^\star \end{pmatrix} \tag{2.1}$$

with non-commuting elements; the deformation (quantization) paramter q is a real number and the star operation is involutional $((a^\star)^\star = a, (b^\star)^\star = b)$. The commutation relations among the (variable) elements $(a, b, a^\star, b^\star)$ are fixed by the conditions

$$\begin{pmatrix} a & qb \\ -b^\star & a^\star \end{pmatrix} \begin{pmatrix} a^\star & -b \\ qb^\star & a \end{pmatrix} = \begin{pmatrix} a^\star & -b \\ qb^\star & a \end{pmatrix} \begin{pmatrix} a & qb \\ -b^\star & a^\star \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\det_q \begin{pmatrix} a & qb \\ -b^\star & a^\star \end{pmatrix} = aa^\star + q^2 bb^\star = 1\,. \tag{2.2}$$

The conditions (2.2) are the q-analogues of the conditions to be satisfied by any classical special unitary matrix, namely, $UU^\dagger = U^\dagger U = I$ and $\det U = 1$. When $q = 1$ the *-operation corresponds to the usual complex conjugation and the quantum $\mathrm{SU}_q(2)$-matrix becomes the classical SU(2)-matrix with $a, b \in \mathbb{C}$. Concrete realizations of the nomcommutative elements of these quantum matrices can also be given (see [4, 21]).

A pair of q-oscillators with $(A_1, A_1^\dagger)$ and $(A_2, A_2^\dagger)$ as the (annihilation, creation) operators is an $\mathrm{SU}_q(2)$-covariant system if

$$\begin{aligned} &A_1A_2 = qA_2A_1 \qquad A_1A_2^\dagger = qA_2^\dagger A_1 \\ &A_1A_1^\dagger - q^2A_1^\dagger A_1 = 1 \\ &A_2A_2^\dagger - q^2A_2^\dagger A_2 = [A_1, A_1^\dagger] \equiv 1 + (q^2-1)A_1^\dagger A_1\,. \end{aligned} \tag{2.3}$$

Throughout, $(\ )^\dagger$ denotes the Hermitian conjugate of ( ). By $SU_q(2)$-covariance of the system (2.3), it is meant that the linear transformations

$$\begin{pmatrix} a & qb \\ -b^\star & a^\star \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} \qquad (A_1^\dagger \ \ A_2^\dagger) \begin{pmatrix} a^\star & -b \\ qb^\star & a \end{pmatrix} = (A_1'^\dagger \ \ A_2'^\dagger) \tag{2.4}$$

lead to the same commutation relations (2.3) for $(A_1', A_1'^\dagger)$ and $(A_2', A_2'^\dagger)$ when the $\mathrm{SU}_q(2)$ matrix elements $(a, b, a^\star, b^\star)$ are assumed to commute with $(A_1, A_1^\dagger)$ and $(A_2, A_2^\dagger)$. It should be noted that the particular coupling between the two modes as specified in (2.3) is completely dictated by the required $\mathrm{SU}_q(2)$-covariance.

Generalization of the above scheme to the n-dimensional case leads to the $\mathrm{SU}_q(n)$-covariant system of oscillators: with q as a real number, $i, j = 1, 2, \ldots, n$

$$A_iA_j = qA_jA_i \qquad i < j \tag{2.5a}$$

$$A_iA_j^\dagger = qA_j^\dagger A_i \qquad i \neq j \tag{2.5b}$$

$$\begin{aligned} &A_1A_1^\dagger - q^2A_1^\dagger A_1 = 1 \\ &A_jA_j^\dagger - q^2A_j^\dagger A_j = [A_{j-1}, A_{j-1}^\dagger] \equiv 1 + (q^2-1)\sum_{i<j} A_i^\dagger A_i \qquad j > 1\,. \end{aligned} \tag{2.5c}$$

The operators $(A_i^\dagger, A_i)$ represent the coordinates and corresponding partial derivatives in a non-commutative hyperplane, or form the q-deformed quantum mechanical phase space [3–6]. The $SU_q(n)$-covariance is retained when q is replaced by q^{-1} provided the ordering of the indices $\{i = 1, 2, \ldots, n\}$ is reversed.

The Fock representation of the multimode oscillator system defined by the relations (2.5) can be easily constructed [5]. Assume the existence of a set of Hermitian number operators $\{N_i\}$ corresponding to the individual modes such that

$$[N_i, A_j] = -\delta_{ij} A_j \qquad [N_i, N_j] = 0. \tag{2.6}$$

Let $|0\rangle = |0, 0, \ldots, 0\rangle$ be the unique ground state of the system, defined by

$$N_i|0\rangle = 0 \qquad A_i|0\rangle = 0 \tag{2.7}$$

and $\{|n_1, n_2, \ldots, n_n\rangle \mid n_1, n_2, \ldots, n_n = 0, 1, 2, \ldots\}$ be the complete set of orthonormal number eigenstates

$$\begin{aligned} &N_i|n_1, n_2, \ldots, n_i, \ldots n_n\rangle = n_i|n_1, n_2, \ldots, n_i, \ldots, n_n\rangle \\ &\langle n_1', n_2', \ldots, n_i', \ldots, n_n'|n_1, n_2, \ldots, n_i, \ldots, n_n\rangle = \prod_{i=1}^{n} \delta_{n_i' n_i}. \end{aligned} \tag{2.8}$$

Then, it follows from the commutation relations (2.5) that one can write, in general, up to phase factors

$$\begin{aligned} &A_j|n_1, n_2, \ldots, n_j, \ldots, n_n\rangle = \{h_j(n_1, n_2, \ldots, n_j)\}^{1/2}|n_1, n_2, \ldots, n_j - 1, \ldots, n_n\rangle \\ &A_j^\dagger|n_1, n_2, \ldots, n_j, \ldots, n_n\rangle \\ &\qquad = \{h_j(n_1, n_2, \ldots, n_j + 1)\}^{1/2}|n_1, n_2, \ldots, n_j + 1, \ldots, n_n\rangle \end{aligned} \tag{2.9}$$

where the h_js satisfy the recursion relations

$$\begin{aligned} &h_1(n_1 + 1) - q^2 h_1(n_1) = 1 \qquad h_1(0) = 0 \\ &h_2(n_1, n_2 + 1) - q^2 h_2(n_1, n_2) = 1 + (q^2 - 1) h_1(n_1) \\ &h_2(n_1 + 1, n_2) = q^2 h_2(n_1, n_2) \qquad h_2(n_1, 0) = 0 \\ &\vdots \\ &h_j(n_1, n_2, \ldots, n_j + 1) - q^2 h_j(n_1, n_2, \ldots, n_j) = 1 + (q^2 - 1) \sum_{i<j} h_i(n_1, n_2, \ldots, n_i) \\ &h_j(n_1, n_2, \ldots, n_i + 1, \ldots, n_j) = q^2 h_j(n_1, n_2, \ldots, n_i, \ldots, n_j) \\ &h_j(n_1, n_2, \ldots, n_{j-1}, 0) = 0. \end{aligned} \tag{2.10}$$

Solving the equations (2.10) recursively one can see that

$$h_j(n_1, n_2, \ldots, n_j) = q^{2\sum_{i<j} n_i} \left(\frac{q^{2n_j} - 1}{q^2 - 1} \right) = q^{2\sum_{i<j} n_i} [n_j]_{q^2} \tag{2.11}$$

the notation $[x]_Q$, in general, will stand for $(Q^x - 1)/(Q - 1)$. Thus, the equations (2.8), (2.9) and (2.11) provide a faithful representation of the commutation relations (2.5) and (2.6). Now, it is seen that for any $k \geqslant 1$

$$\langle n_1', n_2', \ldots, n_j', \ldots, n_n' | (A_j^\dagger)^k A_j^k | n_1, n_2, \ldots, n_j, \ldots, n_n \rangle = \begin{cases} 0 & \text{for} \quad n_j < k \\ q^{2k \sum_{i<j} n_i} ([n_j]!/[n_j - k]!) \prod_{i=1}^{n} \delta_{n_i' n_i} & \text{for} \quad n_j \geqslant k \end{cases} \tag{2.12}$$

where $[n]! = [n][n-1][n-2] \ldots [2][1]$, $[0]! = 1$. Hereafter, for the deformed numbers ($[\ ]_Q$s) the subscript Q, representing the base parameter, will not be explicitly written whenever it is known clearly from the context. The matrix elements of $A_j^k (A_j^\dagger)^k$ are given by

$$\langle n_1', n_2', \ldots, n_j', \ldots, n_n' | A_j^k (A_j^\dagger)^k | n_1, n_2, \ldots, n_j, \ldots, n_n \rangle = q^{2k \sum_{i<j} n_i} ([n_j + k]!/[n_j]!) \prod_{i=1}^{n} \delta_{n_i' n_i} \tag{2.13}$$

for any $k \geqslant 1$. This shows that we can write

$$|n_1, n_2, \ldots, n_j, \ldots, n_n\rangle = \frac{(A_1^\dagger)^{n_1} (A_2^\dagger)^{n_2} \ldots (A_j^\dagger)^{n_j} \ldots (A_n^\dagger)^{n_n}}{([n_1]![n_2]! \ldots [n_n]!)^{1/2}} |0, 0, \ldots, 0\rangle. \tag{2.14}$$

Let us now specialize (2.12) and (2.13) to the case $k = 1$. The result is a set of operator identities

$$A_j A_j^\dagger = q^{2 \sum_{i<j} N_i} \left(\frac{q^{2(N_j+1)} - 1}{q^2 - 1} \right) = q^{2 \sum_{i<j} N_i} [N_j + 1] \tag{2.15a}$$

$$A_j^\dagger A_j = q^{2 \sum_{i<j} N_i} [N_j] \tag{2.15b}$$

$$[A_j, A_j^\dagger] = q^{2 \sum_{i<j} N_i} \tag{2.15c}$$

$$\sum_{i=1}^{j} A_i^\dagger A_i = \left[\sum_{i=1}^{j} N_i \right]. \tag{2.15d}$$

It is in view of the identity (2.15d) that the operator $\sum_{i=1}^{n} A_i^\dagger A_i$ has been called the twisted (total) number operator [5]; in the limit $q \to 1$, $(A_i, A_i^\dagger) \to (b_i, b_i^\dagger)$, the bosonic result is recovered ($\sum_{i=1}^{n} b_i^\dagger b_i =$ the total number operator).

Now, we shall express the number operators $\{N_i\}$ corresponding to the individual modes using the above results. To this end, let us recall that the expression (1.3) for N in the case of the single-mode q-oscillator (1.1) is due to the identity

$$\sum_{k=1}^{n} \frac{(1-Q)^k}{(1-Q^k)} \frac{[n]_Q!}{[n-k]_Q!} = n \qquad n = 1, 2, \ldots \tag{2.16}$$

for any complex number Q [22, 23]. In view of this identity (2.16) (with $Q = q^2$) it follows from (2.12) that

$$\sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}\langle n_1', n_2', \ldots, n_j', \ldots, n_n'|(A_j^\dagger)^k A_j^k q^{-2k\sum_{i<j}N_i}|n_1, n_2, \ldots, n_j, \ldots, n_n\rangle$$
$$= n_j \prod_{i=1}^{n}\delta_{n_i' n_i}. \tag{2.17}$$

In other words, the number operators $\{N_j \mid j = 1, 2, \ldots, n\}$ are uniquely determined by the following formulae

$$\begin{aligned} N_1 &= \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}(A_1^\dagger)^k A_1^k \\ &\vdots \\ N_j &= \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}(A_j^\dagger)^k A_j^k q^{-2k\sum_{i<j}N_i} \qquad j = 2, 3, \ldots, n. \end{aligned} \tag{2.18}$$

One may also check directly, using (2.5), that these N_j satisfy the required relations (2.6). It is seen from (2.5*c*) and (2.15*c*) that we have

$$q^{2\sum_{i<j}N_i} = 1 + (q^2 - 1)\sum_{i<j} A_i^\dagger A_i. \tag{2.19}$$

So, the expression for N_j (2.18) can also be written as

$$N_j = \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}(A_j^\dagger)^k A_j^k \{1 + (q^2-1)\sum_{i<j} A_i^\dagger A_i\}^{-k} \tag{2.20}$$

where $\{\ldots\}^{-k}$ is to be understood as a power series expansion.

It may be noted that the equation (2.19) allows one to write, for $q \neq 0$

$$\sum_{i=1}^{j} N_i = \ln\{1 + (q^2-1)\sum_{i=1}^{j} A_i^\dagger A_i\}/2\ln(q) \qquad j = 1, 2, \ldots, n. \tag{2.21}$$

For the single-mode q-oscillator (1.1) the analogue of (2.19) is given by

$$[\alpha, \alpha^\dagger] = 1 + (q-1)\alpha^\dagger\alpha = q^{\mathrm{N}} \tag{2.22}$$

leading to the result (see [18, 24]) that, for $q \neq 0$

$$\mathrm{N} = \ln\{1 + (q-1)\alpha^\dagger\alpha\}/\ln(q). \tag{2.23}$$

Thus, the equation (2.21) is the extension of (2.23) to the multimode case (2.5).

3. Relation between the $SU_q(n)$-covariant system and the other sets of independent q-oscillators

As already mentioned in the introduction the $SU_q(n)$-covariant system of coupled q-oscillators can be related [12–14] to the various sets of independent q-oscillators through Jordan–Wigner/Klein-like transformations. In this section let us review this relationship briefly using the above framework.

From (2.8), (2.9) and (2.11) it is straightforward to see that if one defines

$$\alpha_j = q^{-\sum_{i<j} N_i} A_j \qquad \mathrm{N}_j = N_j \tag{3.1}$$

then $(\alpha_j, \alpha_j^\dagger)$ satisfy the algebra of independent q-oscillators, with relations of the form (1.1)

$$\begin{aligned} &\alpha_i \alpha_j^\dagger - q^{2\delta_{ij}} \alpha_j^\dagger \alpha_i = \delta_{ij} \qquad \text{or} \qquad \alpha_i \alpha_j^\dagger - \alpha_j^\dagger \alpha_i = \delta_{ij} q^{2\mathrm{N}_j} \\ &[\alpha_i, \alpha_j] = 0 \qquad [\alpha_i, \mathrm{N}_j] = \delta_{ij} \alpha_i \qquad [\mathrm{N}_i, \mathrm{N}_j] = 0. \end{aligned} \tag{3.2}$$

Defining

$$a_j = q^{-(\frac{1}{2}N_j + \sum_{i<j} N_i)} A_j \qquad \mathcal{N}_j = N_j \tag{3.3}$$

one arrives at the algebra

$$\begin{aligned} &a_i a_j^\dagger - q^{\delta_{ij}} a_j^\dagger a_i = \delta_{ij} q^{-\mathcal{N}_j} \qquad \text{or} \qquad a_i a_j^\dagger - q^{-\delta_{ij}} a_j^\dagger a_i = \delta_{ij} q^{\mathcal{N}_j} \\ &[a_i, a_j] = 0 \qquad [a_i, \mathcal{N}_j] = \delta_{ij} a_i \qquad [\mathcal{N}_i, \mathcal{N}_j] = 0 \end{aligned} \tag{3.4}$$

corresponding to the set of independent q-oscillators used in the Jordan–Schwinger-type realizations of various q-deformed Lie algebras [8]. Finally, to obtain the ordinary boson algebra one has to define

$$b_j = q^{-\sum_{i<j} N_i} \left(\frac{N_j + 1}{[N_j + 1]} \right)^{1/2} A_j \tag{3.5}$$

as is clear from the matrix representation of $\{A_j\}$ given above.

It is obvious that the above maps from $\{A_j\}$ to $\{\alpha_j\}, \{a_j\}$ and $\{b_j\}$ are invertible

$$A_j = q^{\sum_{i<j} b_i^\dagger b_i} \left(\frac{[b_j^\dagger b_j + 1]}{b_j^\dagger b_j + 1} \right)^{1/2} b_j \tag{3.6a}$$

$$= q^{(\frac{1}{2}\mathcal{N}_j + \sum_{i<j} \mathcal{N}_i)} a_j \tag{3.6b}$$

$$= q^{\sum_{i<j} \mathrm{N}_i} \alpha_j . \tag{3.6c}$$

It is interesting to see that when $q = -1$ the operators $(A_j, A_j^\dagger)$ correspond to anticommuting bosons, the building blocks in the well-known Green ansatz for constructing parabosons. Then the map $\{b_j\} \to \{A_j\}$ in (3.6a) reduces to

$$A_j(q = -1) = \mathrm{e}^{\mathrm{i}\pi \sum_{i<j} b_i^\dagger b_i} b_j . \tag{3.7}$$

This is precisely the basis of a simple description of a parabose field in terms of a single boson field [25].

4. Coherent states

Regarding the construction of coherent states for the single-mode q-oscillator (1.1) the following observation had been made in [20]. If we write the expression for N given by (1.3) as

$$\mathrm{N} = \alpha^\dagger \mathrm{X} \alpha \qquad \mathrm{X} = \mathrm{X}^\dagger = \sum_{k=1}^{\infty} \frac{(1-q)^k}{(1-q^k)} (\alpha^\dagger)^{k-1} \alpha^{k-1} \tag{4.1}$$

then

$$[\alpha, \alpha^\dagger \mathrm{X}] = 1 \tag{4.2}$$

as is obvious. Consequently

$$\mathrm{e}^{-z\alpha^\dagger \mathrm{X}} \alpha \mathrm{e}^{z\alpha^\dagger \mathrm{X}} = \alpha + z \tag{4.3}$$

and hence the coherent states of the q-oscillator (1.1) may be defined as

$$|z\rangle \sim \mathrm{e}^{z\alpha^\dagger \mathrm{X}} |0\rangle \tag{4.4}$$

up to normalization, such that

$$\alpha |z\rangle = z |z\rangle . \tag{4.5}$$

Now, it is straightforward to extend the above procedure to construct the coherent states of the individual modes of the sytem (2.5). Defining, for each j

$$N_j = A_j^\dagger X_j A_j \qquad X_j = X_j^\dagger = \sum_{k=1}^{\infty} \frac{(1-q^2)^k}{(1-q^{2k})} (A_j^\dagger)^{k-1} A_j^{k-1} q^{-2k \sum_{i<j} N_i} \tag{4.6}$$

we have

$$\mathrm{e}^{-zA_j^\dagger X_j} A_j \mathrm{e}^{zA_j^\dagger X_j} = A_j + z . \tag{4.7}$$

Thus, the states

$$|z_j\rangle \sim \mathrm{e}^{z_j A_j^\dagger X_j} |0\rangle \tag{4.8}$$

are coherent states of the jth mode such that

$$A_i |z_j\rangle = \delta_{ij} z_j |z_j\rangle . \tag{4.9}$$

It is to be noted that since $\{A_j \mid j = 1, 2, \ldots, n\}$ do not commute with each other a state can be an eigenstate of only one of the A_js with a non-zero eigenvalue.

5. Realization in terms of q-derivatives

As is known, the single-mode q-oscillator algebra (1.1) can be realized as

$$\alpha^\dagger\psi(z) = z\psi(z) \qquad \alpha\psi(z) = D_q\psi(z) = \frac{\psi(qz)-\psi(z)}{(q-1)z} = \frac{(q^{z\mathrm{d}/\mathrm{d}z}-1)\psi(z)}{z(q-1)}$$
$$N\psi(z) = z\frac{\mathrm{d}\psi(z)}{\mathrm{d}z} \tag{5.1}$$

where D_q is the Jackson q-derivative operator (see [14, 26]). The operators α and $\alpha^\dagger$ are Hermitian conjugates in this realization when considered in a suitably defined q-Bargmann space [27]. This realization can be extended to the multimode system (2.5) using the map $\{\alpha_j\} \to \{A_j\}$ defined by (3.6*c*). Let $(z_1, z_2, \ldots, z_n)$ be an ordered set of commuting coordinates and $\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\right)$ be the corresponding derivatives. Then, we can define

$$\begin{aligned}
A_j^\dagger\psi(z_1, z_2, \ldots, z_j, \ldots, z_n) &= z_j\psi(qz_1, qz_2, \ldots, qz_{j-1}, z_j, z_{j+1}, \ldots, z_n)\\
A_j\psi(z_1, z_2, \ldots, z_j, \ldots, z_n)&\\
&= \frac{1}{(q^2-1)z_j}\{\psi(qz_1, qz_2, \ldots, qz_{j-1}, q^2z_j, z_{j+1}, \ldots, z_n)\\
&\qquad - \psi(qz_1, qz_2, \ldots, qz_{j-1}, z_j, z_{j+1}, \ldots, z_n)\}\\
N_j\psi(z_1, z_2, \ldots, z_j, \ldots, z_n) &= z_j\frac{\partial}{\partial z_j}\psi(z_1, z_2, \ldots, z_j, \ldots, z_n)
\end{aligned} \tag{5.2}$$

(see also [14]). It can be verified easily that the relations (2.18) are consistent with the definitions (5.2), using the following identity [23]: in the case of the single variable z

$$z\frac{\mathrm{d}\psi(z)}{\mathrm{d}z} = \sum_{k=1}^{\infty}\frac{(1-q)^k}{(1-q^k)}z^k(D_q)^k\psi(z). \tag{5.3}$$

The q-derivative operator defined by (5.1) has been used [28] to deform the one-dimensional Schrödinger equation. The set of ordered q-deformed partial derivatives defined by (5.2) may be useful in experiments with deformed Schrödinger equations in higher dimensions.

6. $\mathrm{GL}_{p,q}(n)$-covariant system of deformed oscillators

The quantum group $\mathrm{SU}_q(n)$ considered above is a special case of $\mathrm{GL}_q(n)$, the q-deformed version of the n-dimensional general linear group. In the deformation of the classical group $\mathrm{GL}(n)$ one can consistently introduce, in general, $\frac{1}{2}n(n-1)+1$ parameters [29, 30]. In this section we shall consider the coupled multimode oscillator systems covariant under $\mathrm{GL}_{p,q}(n)$ with two independent deformation parameters.

A multimode system of oscillators with $\{A_j^+ \mid j = 1, 2, \ldots, n\}$ and $\{A_j^- \mid j =, 1, 2, \ldots, n\}$ as the creation and annihilation operators, or the non-commutative differential calculus with coordinates $\{A^+ \mid j = 1,2,\ldots,n\}$ and derivatives $\{A_j^- \mid j = 1,2,\ldots,n\}$, will have $\mathrm{GL}_{p,q}(n)$-covariance if the following relations are satified

$$
\begin{aligned}
& A_i^- A_j^- = q A_j^- A_i^- \qquad A_i^+ A_j^+ = p^{-1} A_j^+ A_i^+ \qquad i<j \\
& A_i^- A_j^+ = \begin{cases} p A_j^+ A_i^- & \text{if} \quad i<j \\ q A_j^+ A_i^- & \text{if} \quad i>j \end{cases} \\
& A_1^- A_1^+ - pq A_1^+ A_1^- = 1 \\
& A_j^- A_j^+ - pq A_j^+ A_j^- = 1 + (pq-1) \sum_{i<j} A_i^+ A_i^- \qquad j = 2,3,\ldots,n
\end{aligned}
\tag{6.1}
$$

if (p,q) are replaced by (p^{-1},q^{-1}) respectively the orderings $<$ and $>$ are to be interchanged (see [30] for details). In general, the deformation parameters p and q can be independent non-zero complex numbers. When $p = q$ the above system (6.1) becomes $\mathrm{GL}_q(n)$-covariant. Note that (A_j^+, A_j^-) can be Hermitian conjugates of each other only when $p = q^\star$.

First, as an example, we shall consider the two-dimensional case. Let a $\mathrm{GL}_{p,q}(2)$ quantum matrix be written as

$$
T = \begin{pmatrix} a & qb \\ c & d \end{pmatrix}
\tag{6.2}
$$

where the matrix elements are required to obey the commutation relations

$$
\begin{aligned}
& ab = pba \qquad cd = pdc \qquad ac = qca \qquad bd = qdb \\
& pbc = qcb \qquad ad - da = (pq-1)bc.
\end{aligned}
\tag{6.3}
$$

The quantum determinant of T is defined by

$$
\det_{p,q} T = ad - pqbc
\tag{6.4}
$$

and is non-commuting with the elements of T. One can however define an inverse quantum matrix for T

$$
\begin{aligned}
& T^{-1} = (\det_{p,q} T)^{-1} \begin{pmatrix} d & -b \\ -qc & a \end{pmatrix} = \begin{pmatrix} d & -qb/p \\ -pc & a \end{pmatrix} (\det_{p,q} T)^{-1} \\
& TT^{-1} = T^{-1}T = 1
\end{aligned}
\tag{6.5}
$$

(see [31, 32] for more details). Now, it may be checked that for a pair of oscillators the algebraic relations

$$
\begin{aligned}
& A_1^- A_2^- = q A_2^- A_1^- \qquad A_1^+ A_2^+ = p^{-1} A_2^+ A_1^+ \\
& A_1^- A_2^+ = p A_2^+ A_1^- \qquad A_2^- A_1^+ = q A_1^+ A_2^- \\
& A_1^- A_1^+ - pq A_1^+ A_1^- = 1 \\
& A_2^- A_2^+ - pq A_2^+ A_2^- = 1 + (pq-1) A_1^+ A_1^-
\end{aligned}
\tag{6.6}
$$

a special case of (6.1) corresponding to $n=2$, remain invariant under the $\mathrm{GL}_{p,q}(2)$ transformations

$$A_i^- \to \sum_{j=1}^{2} T_{ij} A_j^- \qquad A_i^+ \to \sum_{j=1}^{2} A_j^+ (T^{-1})_{ji} \qquad i=1,2 \tag{6.7}$$

with the assumption that the matrix elements (T_{ij}) commute with $\{A_i^\pm \mid i=1,2\}$. When $p=q$ is a real parameter and $d=a^\star, c=-b^\star$ under a *-involution $\mathrm{GL}_{p,q}(2)$ reduces to $U_q(2)$; then, the quantum determinant is a central element of the algebra. With the further restriction that the quantum determinant $(aa^\star + q^2 bb^\star)$ is equal to 1, $U_q(2) \to \mathrm{SU}_q(2)$.

We shall now briefly indicate the extension of the results discussed for the $\mathrm{SU}_q(n)$-covariant system (2.5) to the case of the $\mathrm{GL}_{p,q}(n)$-covariant system (6.1). As before, let us seek the expressions for a set of independent number operators $\{\tilde{N}_i \mid i=1,2,\ldots,n\}$ in terms of $\{A_i^\pm \mid i=1,2,\ldots,n\}$ such that

$$[\tilde{N}_i, \tilde{N}_j] = 0 \qquad [\tilde{N}_i, A_j^\pm] = \pm\delta_{ij} A_j^\pm. \tag{6.8}$$

Considering the space of number eigenkets $\{|n_1,n_2,\ldots,n_j\ldots,n_n\rangle \mid n_1,n_2,\ldots, n_j,\ldots,n_n = 0,1,2,\ldots\}$ defined by

$$\begin{aligned} &\tilde{N}_j|n_1,n_2,\ldots,n_j,\ldots,n_n\rangle = n_j|n_1,n_2,\ldots,n_j,\ldots,n_n\rangle \\ &A_j^-|n_1,n_2,\ldots,n_{j-1},0,n_{j+1},\ldots,n_n\rangle = 0 \qquad j=1,2,\ldots,n \end{aligned} \tag{6.9}$$

the analogues of the equations in (2.9) and (2.11) read

$$\begin{aligned} &A_j^-|n_1,n_2,\ldots,n_j,\ldots,n_n\rangle = h_j^-(n_1,n_2,\ldots,n_j)|n_1,n_2,\ldots,n_j-1,\ldots,n_n\rangle \\ &A_j^+|n_1,n_2,\ldots,n_j,\ldots,n_n\rangle = h_j^+(n_1,n_2,\ldots,n_j+1)|n_1,n_2,\ldots,n_j+1,\ldots,n_n\rangle \\ &h_j^-(n_1,n_2,\ldots,n_j) = q^{\sum_{i<j} n_i}([n_j]_{pq})^{1/2} = q^{\sum_{i<j} n_i}\left(\frac{(pq)^{n_j}-1}{pq-1}\right)^{1/2} \\ &h_j^+(n_1,n_2,\ldots,n_j) = p^{\sum_{i<j} n_i}([n_j]_{pq})^{1/2}. \end{aligned} \tag{6.10}$$

Starting from the vacuum state $|0,0,\ldots,0\rangle$ one can build the excited states

$$\begin{aligned} &|n_1,n_2,\ldots,n_n\rangle = \frac{(A_1^+)^{n_1}(A_2^+)^{n_2}\ldots(A_n^+)^{n_n}}{([n_1]![n_2]!\ldots[n_n]!)^{1/2}}|0,0,\ldots,0\rangle \\ &n_1,n_2,\ldots,n_n,=0,1,2,\ldots. \end{aligned} \tag{6.11}$$

Let $\langle 0,0,\ldots 0|$ be the dual (bra vector) corresponding to the vacuum ket $|0,0,\ldots,0\rangle$ such that $\langle 0,0,\ldots,0|0,0,\ldots,0\rangle = 1$ and $\langle 0,0,\ldots,0|A_i^+ = 0$ for $i=1,2,\ldots,n$. Then, the set of bra vectors

$$\begin{aligned} &\langle n_1,n_2,\ldots,n_n| = \langle 0,0,\ldots,0|\frac{(A_n^-)^{n_n}(A_{n-1}^-)^{n_{n-1}}\ldots(A_1^-)^{n_1}}{([n_n]![n_{n-1}]!\ldots[n_1]!)^{1/2}} \\ &\qquad n_1,n_2,\ldots,n_n,=0,1,2,\ldots \end{aligned} \tag{6.12}$$

define the dual of the space of number eigenkets (6.11) with

$$\langle n_1', n_2', \ldots, n_n' | n_1, n_2, \ldots, n_n \rangle = \prod_{i=1}^{n} \delta_{n_i' n_i}. \tag{6.13}$$

Now, for the matrix elements of $(A_j^{\pm})^k (A_j^{\mp})^k$ we have

$$\langle n_1', n_2', \ldots, n_j', \ldots, n_n' | (A_j^+)^k (A_j^-)^k | n_1, n_2, \ldots n_j, \ldots, n_n \rangle$$
$$= \begin{cases} 0 & \text{for} \quad n_j < k \\ (pq)^{k \sum_{i<j} n_i} \left([n_j]! / [n_j - k]! \right) \prod_{i=1}^{n} \delta_{n_i' n_i} & \text{for} \quad n_j \geqslant k \end{cases} \tag{6.14a}$$

$$\langle n_1', n_2', \ldots, n_j', \ldots, n_n' | (A_j^-)^k (A_j^+)^k | n_1, n_2, \ldots n_j, \ldots, n_n \rangle$$
$$= (pq)^{k \sum_{i<j} n_i} \left([n_j + k]! / [n_j]! \right) \prod_{i=1}^{n} \delta_{n_i' n_i}. \tag{6.14b}$$

Specializing these equations to the case $k = 1$ we get the operator identites

$$A_j^+ A_j^- = (pq)^{\sum_{i<j} \tilde{N}_i} [\tilde{N}_j] \tag{6.15a}$$

$$A_j^- A_j^+ = (pq)^{\sum_{i<j} \tilde{N}_i} [\tilde{N}_j + 1] \tag{6.15b}$$

$$[A_j^-, A_j^+] = (pq)^{\sum_{i<j} \tilde{N}_i} \tag{6.15c}$$

$$\sum_{i=1}^{n} A_i^+ A_i^- = \left[\sum_{i=1}^{n} \tilde{N}_i \right]. \tag{6.15d}$$

From (6.14a) and the identity (2.16) (now with $Q = pq$) the number operators are identified to be

$$\begin{aligned} \tilde{N}_1 &= \sum_{k=1}^{\infty} \frac{(1-pq)^k}{(1-p^k q^k)} (A_1^+)^k (A_1^-)^k \\ &\vdots \\ \tilde{N}_j &= \sum_{k=1}^{\infty} \frac{(1-pq)^k}{(1-p^k q^k)} (A_j^+)^k (A_j^-)^k (pq)^{-k \sum_{i<j} \tilde{N}_i} \\ &= \sum_{k=1}^{\infty} \frac{(1-pq)^k}{(1-p^k q^k)} (A_j^+)^k (A_j^-)^k \{1 + (pq-1) \sum_{i<j} A_i^+ A_i^- \}^{-k}. \end{aligned} \tag{6.16}$$

The analogue of the formula (2.21) reads

$$\sum_{i=1}^{j} \tilde{N}_i = \ln\{1 + (pq-1) \sum_{i=1}^{j} A_i^+ A_i^- \} / \ln(pq). \tag{6.17}$$

Defining

$$\alpha_j^- = q^{-\sum_{i<j}\tilde{N}_i} A_j^- \qquad \alpha_j^+ = p^{-\sum_{i<j}\tilde{N}_i} A_j^+ \qquad \tilde{\mathrm{N}}_j = \tilde{N}_j. \tag{6.18}$$

One obtains the algebra

$$\begin{aligned} &\alpha_i^-\alpha_j^+ - (pq)^{\delta_{ij}}\alpha_j^+\alpha_i^- = \delta_{ij} \qquad \text{or} \qquad \alpha_i^-\alpha_j^+ - \alpha_j^+\alpha_i^- = \delta_{ij}(pq)^{\tilde{\mathrm{N}}_j} \\ &[\alpha_i^\pm, \alpha_j^\pm] = 0 \qquad [\tilde{\mathrm{N}}_i, \alpha_j^\pm] = \pm\delta_{ij}\alpha_j^\pm \qquad [\tilde{\mathrm{N}}_i, \tilde{\mathrm{N}}_j] = 0. \end{aligned} \tag{6.19}$$

The inverse map $\{\alpha^\pm\} \to \{A^\pm\}$ is

$$A_j^- = q^{\sum_{i<j}\tilde{\mathrm{N}}_i}\alpha_j^- \qquad A_j^+ = p^{\sum_{i<j}\tilde{\mathrm{N}}_i}\alpha_j^+ \qquad \tilde{N}_j = \tilde{\mathrm{N}}_j. \tag{6.20}$$

If we let

$$a_j^- = q^{-\sum_{i<j}\tilde{N}_i} p^{-\frac{1}{2}\tilde{N}_j} A_j^- \qquad a_j^+ = A_j^+ p^{-(\frac{1}{2}\tilde{N}_j + \sum_{i<j}\tilde{N}_i)} \qquad \tilde{\mathcal{N}}_j = \tilde{N}_j \tag{6.21}$$

then

$$a_j^- a_j^+ - q^{\delta_{ij}} a_j^+ a_j^- = \delta_{ij} p^{-\tilde{\mathcal{N}}_j} \qquad \text{or} \qquad a_j^- a_j^+ - p^{-\delta_{ij}} a_j^+ a_j^- = \delta_{ij} q^{\tilde{\mathcal{N}}_j} \tag{6.22}$$

$$[a_i^\pm, a_j^\pm] = 0 \qquad [\tilde{\mathcal{N}}_i, a_j^\pm] = \pm\delta_{ij} a_j^\pm \qquad [\tilde{\mathcal{N}}_i, \tilde{\mathcal{N}}_j] = 0. \tag{6.23}$$

The single-mode oscillator of this type (6.22), called (p, q)-oscillator, was introduced in [33] to study the Jordan–Schwinger type realizations of two-parameter (p, q) quantum algebras (see also [34]). There can also be further generalizations [35]. It may be noted that the (p, q)-oscillator gives a unified description of all types of q-oscillators: the q-oscillator (1.1) (or (3.2)) corresponds to $p = 1$, when $p = q$ the algebra (3.4) is obtained, when $pq = -1$ one obtains the q-fermion algebra derivable by contraction from the quantum super algebra $sl_q(1|1)$ [10], and when $p/q = -1$ the resulting q-fermionic algebra [11] (see also [36]) is derivable from the quantum superalgebra $osp_q(2|1)$ [24]. For the single-mode (p, q)-oscillator the number operator has been discussed in [23]. The inverse map $\{a^\pm\} \to \{A^\pm\}$ is

$$A_j^- = p^{\frac{1}{2}\tilde{\mathcal{N}}_j} q^{\sum_{i<j}\tilde{\mathcal{N}}_i} a_j^- \qquad A_j^+ = a_j^+ p^{(\frac{1}{2}\tilde{\mathcal{N}}_j + \sum_{i<j}\tilde{\mathcal{N}}_i)} \qquad \tilde{N}_j = \tilde{\mathcal{N}}_j. \tag{6.24}$$

The relation of $\{A_j^\pm\}$ to the boson algebra follows from the matrix representation (6.10)

$$b_j = q^{-\sum_{i<j}\tilde{N}_i}\left(\frac{\tilde{N}_j + 1}{[\tilde{N}_j + 1]}\right)^{1/2} A_j^- \qquad b_j^\dagger = p^{-\sum_{i<j}\tilde{N}_i} A_j^+ \left(\frac{\tilde{N}_j + 1}{[\tilde{N}_j + 1]}\right)^{1/2} \tag{6.25}$$

with the inverse map

$$A_j^- = q^{\sum_{i<j} b_i^\dagger b_i}\left(\frac{[b_j^\dagger b_j + 1]}{b_j^\dagger b_j + 1}\right)^{1/2} b_j \qquad A_j^+ = p^{\sum_{i<j} b_i^\dagger b_i} b_j^\dagger \left(\frac{[b_j^\dagger b_j + 1]}{b_j^\dagger b_j + 1}\right)^{1/2}. \tag{6.26}$$

The analogue of the realization (5.2) in this case is the following

$$\begin{aligned}
A_j^+\psi(z_1,z_2,\ldots,z_j,\ldots,z_n) &= z_j\psi(pz_1,pz_2,\ldots,pz_{j-1},z_j,z_{j+1},\ldots,z_n)\\
A_j^-\psi(z_1,z_2,\ldots,z_j,\ldots,z_n)&\\
&= \frac{1}{(pq-1)z_j}\{\psi(qz_1,qz_2,\ldots,qz_{j-1},pqz_j,z_{j+1},\ldots,z_n)\\
&\quad -\psi(qz_1,qz_2,\ldots,qz_{j-1},z_j,z_{j+1},\ldots,z_n)\}\\
\tilde{N}_j\psi(z_1,z_2,\ldots,z_j,\ldots,z_n) &= z_j\frac{\partial}{\partial z_j}\psi(z_1,z_2,\ldots,z_j,\ldots,z_n)
\end{aligned}\tag{6.27}$$

where $(z_1,z_2,\ldots,z_n)$ form an ordered set of commuting coordinates. As already noted, the (A^+,A^-)s are Hermitian conjugate pairs only when $p=q^\star$. In this case the (α^+,α^-)s also satisfy the relation $\alpha^+=(\alpha^-)^\dagger$. For a_j^+ and a_j^- defined by (6.21) to be Hermitian conjugates of each other the condition is that either p and q are real or $pq^\star=1$. The bosonic operators synthesized from the $A^\pm$s as in (6.25) are seen to satisfy the Hermiticity requirement $b^\dagger=(b)^\dagger$ in the representation (6.10) independent of the values of p and q.

7. Multimode q-oscillator systems covariant under the quantum supergroups $\mathrm{SU}_q(n|m)$ and $\mathrm{GL}_{p,q}(n|m)$

The systems studied so far are bosonic in the sense that the occcupation number of any level is unlimited. The usual fermionic systems for which the occupation number of any level cannot exceed one, by the Pauli principle, can also be q-deformed: the twisted canonical anticommutation relations [7] read, for an n-level system

$$\begin{aligned}
&\beta_k\beta_l+q\beta_l\beta_k=0 \qquad k<l \qquad \beta_k^\dagger\beta_l^\dagger+q\beta_l^\dagger\beta_k^\dagger=0 \qquad k>l\\
&\beta_k\beta_l^\dagger+q\beta_l^\dagger\beta_k=0 \qquad k\neq l \qquad \beta_1\beta_1^\dagger+\beta_1^\dagger\beta_1=1\\
&\beta_l\beta_l^\dagger+q\beta_l^\dagger\beta_l=1+(q^2-1)\sum_{k<l}\beta_k^\dagger\beta_k \qquad l>1\\
&\qquad\qquad \beta_k^2=\beta_k^{\dagger 2}=0 \qquad k,l=1,2,\ldots,n
\end{aligned}\tag{7.1}$$

with $q\in\mathbf{R}$. These relations possess the $\mathrm{SU}_q(n)$-covariance, exactly in the same way as (2.5); in the limit $q\to1$ the system (7.1) becomes the (classical) $\mathrm{SU}(n)$-covariant fermionic system. As in the q-bosonic case (2.5), if q is replaced by q^{-1} in (7.1) then the ordering of the indices $\{1,2,\ldots,n\}$ is to be reversed to retain the $\mathrm{SU}_q(n)$-covariance.

By combining these q-deformed canonical anticommutation relations (7.1) with the q-deformed canonical commutation relations (2.5) one gets systems of oscillators covariant under the quantum supergroups $\{\mathrm{SU}_q(n|m)\}$. Thus an $\mathrm{SU}_q(n|m)$-covariant system consists of n q-bosonic(A) and m q-fermionic(B) oscillators obeying the following relations with $q \in \mathbf{R}$

$$
\begin{aligned}
& A_i A_j - q A_j A_i = 0 \qquad i < j \qquad A_i A_j^\dagger - q A_j^\dagger A_i = 0 \qquad i \neq j \\
& A_1 A_1^\dagger - q^2 A_1^\dagger A_1 = 1 \qquad A_j A_j^\dagger - q^2 A_j^\dagger A_j = 1 + (q^2 - 1) \sum_{i<j} A_i^\dagger A_i \qquad j > 1 \\
& A_j B_k - q B_k A_j = 0 \qquad A_j B_k^\dagger - q B_k^\dagger A_j = 0 \\
& B_k B_l + q B_l B_k = 0 \qquad k < l \qquad B_k^\dagger B_l^\dagger + q B_l^\dagger B_k^\dagger = 0 \qquad k > l \\
& B_k B_l^\dagger + q B_l^\dagger B_k = 0 \qquad k \neq l \qquad B_k^2 = (B_k^\dagger)^2 = 0 \\
& B_1 B_1^\dagger + B_1^\dagger B_1 = 1 + (q^2 - 1) \sum_{i=1}^{n} A_i^\dagger A_i \\
& B_l B_l^\dagger + B_l^\dagger B_l = 1 + (q^2 - 1) \sum_{i=1}^{n} A_i^\dagger A_i + (q^2 - 1) \sum_{k<l} B_k^\dagger B_k \qquad l > 1 \\
& \qquad i, j = 1, 2, \ldots, n \qquad k, l = 1, 2, \ldots, m\,.
\end{aligned}
\tag{7.2}
$$

For example, when $n = m = 1$ in (7.2) we get a supersymmetric pair of q-oscillators (A, B) obeying the relations

$$
\begin{aligned}
& AB = qBA \qquad AB^\dagger = qB^\dagger A \qquad B^2 = (B^\dagger)^2 = 0 \\
& AA^\dagger - q^2 A^\dagger A = 1 \qquad BB^\dagger + B^\dagger B = 1 + (q^2 - 1) A^\dagger A
\end{aligned}
\tag{7.3}
$$

which remain invariant under the $\mathrm{SU}_q(1|1)$ transformations

$$
\begin{aligned}
& \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} a & \beta \\ -(a^\star)^{-1}\beta^\star(a^\star)^{-1} & (a^\star)^{-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \\
& a\beta = q\beta a \qquad a^\star\beta = q^{-1}\beta a^\star \qquad \beta\beta^\star + q^2\beta^\star\beta = 0 \qquad \beta^2 = (\beta^\star)^2 = 0 \\
& aa^\star - a^\star a = (q^2 - 1)\beta^\star\beta \qquad aa^\star + \beta\beta^\star = 1 \\
& (A'^\dagger \ B'^\dagger) = (A^\dagger \ B^\dagger) \begin{pmatrix} a^\star & -a^{-1}\beta a^{-1} \\ \beta^\star & a^{-1} \end{pmatrix}.
\end{aligned}
\tag{7.4}
$$

Here, it is assumed that $(a, a^\star)$ commute with $(A, A^\dagger)$ and $(B, B^\dagger)$, and $(\beta, \beta^\star)$ commute with $(A, A^\dagger)$ and anticommute with $(B, B^\dagger)$.

Now, the construction of the Fock space for the system (7.2) is obtained by extending the results of [7] for the case (7.1). To this end, we shall take the complete set of orthonormal eigenstates of n bosonic number operators $\{N_j \mid j = 1, 2, \ldots, n\}$

and m fermionic number operators $\{M_l \mid l = 1,2,\ldots,m\}$ to be defined by the relations

$$\begin{aligned}
&N_j|n_1,n_2,\ldots,n_j,\ldots n_n;m_1,m_2,\ldots,m_m\rangle \\
&\qquad = n_j|n_1,n_2,\ldots,n_j,\ldots,n_n;m_1,m_2,\ldots,m_m\rangle \\
&\qquad\qquad n_j = 0,1,2,\ldots \qquad j=1,2,\ldots,n \\
&M_l|n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l,\ldots,m_m\rangle \\
&\qquad = m_l|n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l,\ldots,m_n\rangle \\
&\qquad\qquad m_l = 0,1 \qquad l=1,2,\ldots,m \\
&A_j|0,0,\ldots,0;0,0,\ldots,0\rangle = 0 \qquad B_l|0,0,\ldots,0;0,0,\ldots,0\rangle = 0 \\
&\langle n_1',n_2',\ldots,n_n';m_1',m_2',\ldots,m_m'|n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_m\rangle \\
&\qquad = \prod_{i=1}^{n}\delta_{n_i'n_i}\prod_{j=1}^{n}\delta_{m_j'm_j}.
\end{aligned} \tag{7.5}$$

The number operators must satisfy the commutation relations

$$\begin{aligned}
&\left[N_i, A_j^\dagger\right] = \delta_{ij}A_j^\dagger \qquad \left[M_k, B_l^\dagger\right] = \delta_{kl}B_l^\dagger \\
&[N_i,N_j] = 0 \qquad [N_i,M_k] = 0 \qquad [M_k,M_l] = 0.
\end{aligned} \tag{7.6}$$

It is found that one can write, up to phase factors

$$\begin{aligned}
&A_j|n_1,n_2,\ldots,n_j,\ldots n_n;m_1,m_2,\ldots,m_m\rangle \\
&\qquad = h_j^-(n_1,n_2,\ldots,n_j)|n_1,n_2,\ldots,n_j-1,\ldots,n_n;m_1,m_2,\ldots,m_m\rangle \\
&A_j^\dagger|n_1,n_2,\ldots,n_j,\ldots n_n;m_1,m_2,\ldots,m_m\rangle \\
&\qquad = h_j^+(n_1,n_2,\ldots,n_j+1)|n_1,n_2,\ldots,n_j+1,\ldots,n_n;m_1,m_2,\ldots,m_m\rangle \\
&B_l|n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l,\ldots,m_m\rangle \\
&\qquad = g_l^-(n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l) \\
&\qquad\quad \times|n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l-1,\ldots,m_m\rangle \\
&B_l^\dagger|n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l,\ldots,m_m\rangle \\
&\qquad = g_l^+(n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l) \\
&\qquad\quad \times|n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l+1,\ldots,m_m\rangle \\
&h_j^-(n_1,n_2,\ldots,n_j) = h_j^+(n_1,n_2,\ldots,n_j) = q^{\sum_{i<j}n_i}\left([n_j]_{q^2}\right)^{\frac{1}{2}} \\
&g_l^-(n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l) = q^{\sum_{i=1}^{n}n_i}(-q)^{\sum_{k<l}m_k}m_l \\
&g_l^+(n_1,n_2,\ldots,n_n;m_1,m_2,\ldots,m_l) = q^{\sum_{i=1}^{n}n_i}(-q)^{\sum_{k<l}m_k}(1-m_l).
\end{aligned} \tag{7.7}$$

Further, we have

$$
\begin{aligned}
&|n_1, n_2, \ldots, n_n; m_1, m_2, \ldots, m_m\rangle \\
&\quad = \frac{(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}\ldots(A_n^\dagger)^{n_n}(B_1^\dagger)^{m_1}(B_2^\dagger)^{m_2}\ldots(B_m^\dagger)^{m_m}}{([n_1]![n_2]!\ldots[n_n]!)^{1/2}} \\
&\qquad \times |0,0,\ldots,0,\ldots,0\rangle \\
&\qquad n_1, n_2, \ldots, n_n = 0,1,2,\ldots \qquad m_1, m_2, \ldots, m_m = 0,1
\end{aligned}
\tag{7.8}
$$

with the corresponding bra vectors given by $|\ldots\rangle^\dagger$s. Now, the matrix elements of $(A_j^\dagger)^s(A_j)^s$ and $B_l^\dagger B_l$ are given by

$$
\begin{aligned}
&\langle n_1', \ldots, n_j', \ldots, n_n'; m_1', \ldots, m_m'|(A_j^\dagger)^s(A_j)^s|n_1, \ldots, n_j, \ldots, n_n; m_1, \ldots, m_m\rangle \\
&\quad = \begin{cases} 0 & \text{for} \quad n_j < s \\ q^{2s\sum_{i<j} n_i}([n_j]!/[n_j - s]!)\prod_{i=1}^{n}\delta_{n_i'n_i}\prod_{j=1}^{m}\delta_{m_j'm_j} & \text{for} \quad n_j \geqslant s \end{cases}
\end{aligned}
\tag{7.9a}
$$

$$
\begin{aligned}
&\langle n_1', \ldots, n_n'; m_1', \ldots, m_l', \ldots, m_m'|B_l^\dagger B_l|n_1, \ldots, n_n; m_1 \ldots, m_l, \ldots, m_m\rangle \\
&\quad = \prod_{i=1}^{n}\delta_{n_i'n_i}\prod_{j=1}^{m}\delta_{m_j'm_j}q^{2(\sum_{i=1}^{n} n_i + \sum_{k<l} m_k)}m_l.
\end{aligned}
\tag{7.9b}
$$

Comparing (7.9*a*) with (2.12) it is clear that for the bosonic modes (A) the number operators are given by the same expressions in (2.18)

$$
\begin{aligned}
N_1 &= \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}(A_1^\dagger)^k A_1^k \\
&\vdots \\
N_j &= \sum_{k=1}^{\infty}\frac{(1-q^2)^k}{(1-q^{2k})}(A_j^\dagger)^k A_j^k q^{-2k\sum_{i<j} N_i} \qquad j = 2, \ldots, n.
\end{aligned}
\tag{7.10}
$$

From (7.9*b*) it follows that, for the fermionic modes (B) the number operators are given by

$$
\begin{aligned}
M_1 &= B_1^\dagger B_1 q^{-2\sum_{i=1}^{n} N_i} \\
&\vdots \\
M_l &= B_l^\dagger B_l q^{-2(\sum_{i=1}^{n} N_i + \sum_{k<l} M_k)} \qquad l = 2, \ldots, m.
\end{aligned}
\tag{7.11}
$$

Since m_l has only the two values $(0,1)$, in the Fock space (7.7) one has the identity $[M_l] = M_l$, $[1 - M_l] = 1 - M_l$. With this, the extension of the identities (2.15) to the fermionic sector (B) follows

$$B_l^\dagger B_l = q^{2(\sum_{i=1}^n N_i + \sum_{k<l} M_k)}[M_l] \tag{7.12a}$$

$$B_l B_l^\dagger = q^{2(\sum_{i=1}^n N_i + \sum_{k<l} M_k)}[1 - M_l] \tag{7.12b}$$

$$B_l B_l^\dagger + B_l^\dagger B_l = q^{2(\sum_{i=1}^n N_i + \sum_{k<l} M_k)} \tag{7.12c}$$

$$\sum_{k=1}^{l} B_k^\dagger B_k = q^{2\sum_{i=1}^n N_i}\left[\sum_{k=1}^{l} M_k\right] \tag{7.12d}$$

$$\sum_{i=1}^{n} A_i^\dagger A_i + \sum_{k=1}^{l} B_k^\dagger B_k = \left[\sum_{i=1}^{n} N_i + \sum_{k=1}^{l} M_k\right]. \tag{7.12e}$$

It is clear from the matrix representation of the $(B, B^\dagger)$s provided by (7.7) that if we define

$$f_l = q^{-(\sum_{i=1}^n N_i + \sum_{k<l} M_k)} B_l \tag{7.13}$$

then the $(f_l, f_l^\dagger)$s satisfy the ordinary fermion algebra

$$f_k f_l^\dagger + f_l^\dagger f_k = \delta_{kl} \qquad f_k f_l + f_l f_k = 0. \tag{7.14}$$

Defining

$$F_l = q^{-\{\sum_{i=1}^n N_i + \sum_{k<l} M_k + (M_l/2)\}} B_l \qquad \mathcal{M}_l = M_l \tag{7.15}$$

one gets a system of anticommuting q-fermions with the algebra

$$F_k F_l^\dagger + q^{-\delta_{kl}} F_l^\dagger F_k = q^{-\mathcal{M}_k}\delta_{kl} \tag{7.16a}$$

$$F_k F_l + F_l F_k = 0 \qquad [\mathcal{M}_k, F_l^\dagger] = \delta_{kl} F_l^\dagger. \tag{7.16b}$$

The inverse maps $(f, F) \rightarrow (B)$ are easily constructed. These relations between the various q-fermionic systems are the same as in [13]. In the single-mode situation the q-fermion algebra (7.16) can be obtained from an Inönu–Wigner contraction of the quantum superalgebra $sl_q(1|1)$ [10]. It should be noted that there exists another single-mode q-fermionic algebra in which one would have $q^{+\mathcal{M}_k}$ instead of $q^{-\mathcal{M}_k}$ in the right hand side of (7.16a); this algebra is obtainable from the quantum superalgebra $osp_q(2|1)$ (see [24, 36] for further details). As already mentioned these two algebras are the two special cases $(pq = -1, p/q = -1)$ of the (p, q)-oscillator algebra defined in section-6. Further, it is interesting to note that when $q = -1$ the twisted canonical anticommutation relations (7.1) represent the algebra of commuting fermions which are the building blocks to obtain the parafermi algebra using the Green ansatz. It may be noted that the coherent states of the bosonic(A) modes are given by the same expressions as in section 4. The coherent states of the fermionic(B) modes can be constructed by a straightforward extension of the standard procedure, using Grassmann variables, employed in the case of ordinary fermions

$$|\theta_l\rangle \sim (|0\rangle - \theta_l B_l^\dagger|0\rangle) \qquad B_k|\theta_l\rangle = \delta_{kl}\theta_l|\theta_l\rangle \tag{7.17}$$

where the θs are Grassmann variables anticommuting with the $(B, B^\dagger)$s; (see [37] for details).

Inspired by the relation (7.13) one can write down a realization of the (A, B)-operators obeying (7.2) analogous to the realization of the A-operators in section 5. To this end, we must now consider the functions $\psi(z_1, \ldots, z_n; \theta_1, \ldots, \theta_m)$ where $(z_1, z_2, \ldots, z_n)$ are ordered commuting variables, as before, and $(\theta_1, \theta_2, \ldots, \theta_m)$ form an ordered set of Grassmann variables $(\theta_k\theta_l + \theta_l\theta_k = 0)$. Then, in the space of these functions (Grassmann algebra formed by linear combination of monomials in θs with the coefficients being functions of zs (see [38]))

$$
\begin{aligned}
&A_j^\dagger \psi(z_1, z_2, \ldots, z_j, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad = z_j \psi(qz_1, qz_2, \ldots, qz_{j-1}, z_j, \ldots, z_n; \theta_1, \theta_2, \ldots \theta_m) \\
&A_j \psi(z_1, z_2, \ldots, z_j, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad = \frac{1}{(q^2-1)z_j}\{\psi(qz_1, qz_2, \ldots, qz_{j-1}, q^2 z_j, z_{j+1}, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad\quad - \psi(qz_1, qz_2, \ldots, qz_{j-1}, z_j, z_{j+1}, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m)\} \\
&B_l^\dagger \psi(z_1, z_2, \ldots, z_n; \theta_1, \theta_2, \ldots \theta_l, \ldots, \theta_m) \\
&\qquad = \theta_l \psi(qz_1, qz_2, \ldots, qz_n; q\theta_1, q\theta_2, \ldots, q\theta_{l-1}, \theta_l, \ldots, \theta_m) \\
&B_l \psi(z_1, z_2, \ldots, z_n; \theta_1, \theta_2, \ldots \theta_l, \ldots, \theta_m) \\
&\qquad = \frac{\partial}{\partial\theta_l}\psi(qz_1, qz_2, \ldots, qz_n, ; q\theta_1, q\theta_2, \ldots, q\theta_{l-1}, \theta_l, \ldots, \theta_m) \\
&N_j\psi = z_j \frac{\partial\psi}{\partial z_j} \qquad M_l\psi = \theta_l \frac{\partial\psi}{\partial\theta_l}.
\end{aligned}
\tag{7.18}
$$

The two-parameter generalization of the above results can be obtained in a straightforward manner. A pair of deformed supersymmetric oscillators $(A^\pm, B^\pm)$ will have $\mathrm{GL}_{p,q}(1|1)$-covariance if the following relations are satisfied

$$
\begin{aligned}
&A^-B^- = qB^-A^- \qquad A^+B^+ = p^{-1}B^+A^+ \\
&A^-B^+ = pB^+A^- \qquad A^+B^- = q^{-1}B^-A^+ \\
&A^-A^+ - pqA^+A^- = 1 \qquad (B^\pm)^2 = 0 \\
&B^-B^+ + B^+B^- = 1 + (pq-1)A^+A^-.
\end{aligned}
\tag{7.19}
$$

The $\mathrm{GL}_{pq}(1|1)$-transformations leaving invariant these relations are given by

$$
\begin{aligned}
&\begin{pmatrix} A^- \\ B^- \end{pmatrix} \to \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}\begin{pmatrix} A^- \\ B^- \end{pmatrix} \qquad (A^+B^+) \to (A^+B^+)\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}^{-1} \\
&a\beta = p\beta a \qquad a\gamma = q\gamma a \qquad d\beta = p\beta d \qquad d\gamma = q\gamma d \\
&\beta\gamma = -(q/p)\gamma\beta \qquad ad - da = (q - p^{-1})\gamma\beta \qquad \beta^2 = \gamma^2 = 0 \\
&\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} + d^{-1}\beta a^{-1}\gamma a^{-1} & -a^{-1}\beta d^{-1} \\ -d^{-1}\gamma a^{-1} & a^{-1} - a^{-1}\beta d^{-1}\gamma d^{-1} \end{pmatrix}
\end{aligned}
\tag{7.20}
$$

where $p, q \in \mathbf{C}/\{0\}$ (see [32, 39, 40] , for more details). In the higher dimensional generalization one would have n (A^+, A^-)-pairs and m (B^+, B^-)-pairs obeying the relations

$$
\begin{aligned}
& A_i^- A_j^- = q A_j^- A_i^- \qquad A_i^+ A_j^+ = p^{-1} A_j^+ A_i^+ \qquad i < j \\
& A_i^- A_j^+ = \begin{cases} p A_j^+ A_i^- & \text{if} \quad i < j \\ q A_j^+ A_i^- & \text{if} \quad i > j \end{cases} \\
& A_1^- A_1^+ - pq A_1^+ A_1^- = 1 \\
& A_j^- A_j^+ - pq A_j^+ A_j^- = 1 + (pq - 1) \sum_{i<j} A_i^+ A_i^- \qquad j = 2, 3, \ldots, n \\
& A_j^- B_k^- - q B_k^- A_j^- = 0 \qquad A_j^- B_k^+ - p B_k^+ A_j^- = 0 \\
& A_j^+ B_k^+ - p^{-1} B_k^+ A_j^+ = 0 \qquad A_j^+ B_k^- - q^{-1} B_k^- A_j^+ = 0 \\
& B_k^- B_l^- + q B_l^- B_k^- = 0 \qquad k < l \qquad B_k^+ B_l^+ + p B_l^+ B_k^+ = 0 \qquad k > l \\
& B_k^- B_l^+ + p B_l^+ B_k^- = 0 \qquad k < l \qquad B_k^- B_l^+ + q B_l^+ B_k^- = 0 \qquad k > l \\
& (B_k^\pm)^2 = 0 \qquad B_1^- B_1^+ + B_1^+ B_1^- = 1 + (pq - 1) \sum_{i=1}^{n} A_i^+ A_i^- \\
& B_l^- B_l^+ + B_l^+ B_l^- = 1 + (pq - 1) \sum_{i=1}^{n} A_i^+ A_i^- + (pq - 1) \sum_{k<l} B_k^+ B_k^- \\
& \qquad l = 2, 3, \ldots, n
\end{aligned} \tag{7.21}
$$

which are left invariant under the $\mathrm{GL}_{p,q}(n|m)$ transformations.

Let us define, analogous to (7.8)

$$
\begin{aligned}
& |n_1, n_2, \ldots, n_n; m_1, m_2, \ldots, m_m\rangle \\
& \qquad = \frac{(A_1^+)^{n_1} (A_2^+)^{n_2} \ldots (A_n^+)^{n_n} (B_1^+)^{m_1} (B_2^+)^{m_2} \ldots (B_m^+)^{m_m}}{([n_1]! [n_2]! \ldots [n_n]!)^{\frac{1}{2}}} \\
& \qquad \quad \times |0, 0, \ldots, 0; 0, \ldots, 0\rangle \\
& n_1, n_2, \ldots, n_n = 0, 1, 2, \ldots \qquad m_1, m_2, \ldots, m_m = 0, 1 \qquad [n] = [n]_{pq}
\end{aligned} \tag{7.22}
$$

and the corresponding vectors of the dual space

$$
\begin{aligned}
& \langle n_1, n_2, \ldots, n_n; m_1, m_2, \ldots, m_m| \\
& \qquad = \langle 0, 0, \ldots, 0; 0, \ldots, 0| \\
& \qquad \quad \times \frac{(B_m^-)^{m_m} (B_{m-1}^-)^{m_{m-1}} \ldots (B_1^-)^{m_1} (A_n^-)^{n_n} \ldots (A_1^-)^{n_1}}{([n_1]! [n_2]! \ldots [n_n]!)^{1/2}}.
\end{aligned} \tag{7.23}
$$

In the space of these vectors the matrix representations of $(A^{\pm}, B^{\pm})$ can be worked out using the relations (7.21). Now, the number operators are given by the expressions in (6.16) for $\{\tilde{N}_j \mid j = 1, 2, \ldots, n\}$ associated with the n bosonic(A) modes and

$$\begin{aligned}
\tilde{M}_1 &= B_1^+ B_1^- (pq)^{-\sum_{i=1}^n \tilde{N}_i} \\
&\vdots \\
\tilde{M}_l &= B_l^+ B_l^- (pq)^{-(\sum_{i=1}^n \tilde{N}_i + \sum_{k<l} \tilde{M}_k)} \qquad l = 2, 3, \ldots, m
\end{aligned} \tag{7.24}$$

for the m fermionic(B) modes. The set of operator identities (7.12) are generalized by the replacement of q^2 by pq. Extension of the realization (7.18) to the case of $(A^{\pm}, B^{\pm})$ is obtained by defining

$$\begin{aligned}
& A_j^+ \psi(z_1, z_2, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad = z_j \psi(pz_1, pz_2, \ldots, pz_{j-1}, z_j, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
& A_j^- \psi(z_1, z_2, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad = \frac{1}{(pq-1)z_j} \\
&\qquad\quad \times \{\psi(qz_1, qz_2, \ldots, qz_{j-1}, pqz_j, z_{j+1}, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad\quad - \psi(qz_1, qz_2, \ldots, qz_{j-1}, z_j, z_{j+1}, \ldots, z_n; \theta_1, \theta_2, \ldots \theta_m)\} \\
& B_l^+ \psi(z_1, z_2, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad = \theta_l \psi(pz_1, pz_2, \ldots, pz_n; p\theta_1, p\theta_2, \ldots, p\theta_{l-1}, \theta_l, \ldots, \theta_m) \\
& B_l^- \psi(z_1, z_2, \ldots, z_n; \theta_1, \theta_2, \ldots, \theta_m) \\
&\qquad = \frac{\partial}{\partial \theta_l} \psi(qz_1, qz_2, \ldots, qz_n; q\theta_1, \ldots, q\theta_{l-1}, \theta_l, \ldots, \theta_m) \\
& N_j \psi = z_j \frac{\partial \psi}{\partial z_j} \qquad M_l \psi = \theta_l \frac{\partial \psi}{\partial \theta_l}.
\end{aligned} \tag{7.25}$$

The matrix representation of the $(A^{\pm}, B^{\pm})$s can be obtained using (7.24) and noting the correspondence

$$|n_1, n_2, \ldots, n_n; m_1, m_2, \ldots, m_m\rangle \leftrightarrow \frac{z_1^{n_1} z_2^{n_2} \ldots z_n^{n_n} \theta_1^{m_1} \theta_2^{m_2} \ldots \theta_m^{m_m}}{([n_1]![n_2]! \ldots [n_n]!)^{1/2}}. \tag{7.26}$$

8. Realization of classical groups in non-commutative spaces

Based on the discussion in sections 3 and 4, let us now observe that classical groups can be realized in non-commutative spaces. Following (3.1) and (4.6) define

$$\begin{aligned}
\alpha_j^\dagger &= q^{-\sum_{i<j} N_i} A_j^\dagger \\
\mathcal{A}_j &= X_j A_j q^{\sum_{i<j} N_i} \\
&= \sum_{k=1}^{\infty} \frac{(1-q^2)^k}{(1-q^{2k})} (A_j^+)^{k-1} A_j^k q^{-(2k-1)\sum_{i<j} N_i}
\end{aligned} \tag{8.1}$$

and note that $N_j = \alpha_j^\dagger \mathcal{A}_j$. Then, it is easy to check that

$$[\mathcal{A}_j, \alpha_k^\dagger] = \delta_{jk} \qquad [\mathcal{A}_j, \mathcal{A}_k] = [\alpha_j^\dagger, \alpha_k^\dagger] = 0. \tag{8.2}$$

Now, let us observe that

$$\xi_{jk} = \alpha_j^\dagger \mathcal{A}_k \qquad j, k = 1, 2, \ldots, n \tag{8.3}$$

obey the classical $gl(n)$ algebra

$$[\xi_{jk}, \xi_{lm}] = \delta_{kl}\xi_{jm} - \delta_{jm}\xi_{lk}. \tag{8.4}$$

Then, for any classical Lie algebra one can construct the Jordan–Schwinger-type realization by replacing the boson operators $(b_j^\dagger b_k)$ by (ξ_{jk}) respectively; since the ξ_{jk}s are not Hermitian conjugates of ξ_{kj}s we are led to non-Hermitian realizations. The interesting aspect of this realization is that the (ξ_{jk}) act in a space with non-commutative coordinates $(A_j^\dagger)$ and derivatives (A_j) obeying the commutation relations (2.5). In the limit $q \to 1$ this realization reduces to the regular Jordan–Schwinger realization. It may be noted that one can also define

$$\alpha_j = q^{-\sum_{i<j} N_i} A_j \qquad N_j = \mathcal{A}_j^\dagger \alpha_j \qquad \xi_{jk} = \mathcal{A}_j^\dagger \alpha_k \tag{8.5}$$

instead of (8.3) to obtain a similar realization. For example, a realization of the $sl(2)$ algebra is given by

$$X^+ = \alpha_1^\dagger \mathcal{A}_2 \qquad X^- = \alpha_2^\dagger \mathcal{A}_1 \qquad H = \tfrac{1}{2}(\alpha_1^\dagger \mathcal{A}_1 - \alpha_2^\dagger \mathcal{A}_2) = \tfrac{1}{2}(N_1 - N_2) \tag{8.6}$$

or

$$X^+ = \mathcal{A}_1^\dagger \alpha_2 \qquad X^- = \mathcal{A}_2^\dagger \alpha_1 \qquad H = \tfrac{1}{2}(\mathcal{A}_1^+ \alpha_1 - \mathcal{A}_2^+ \alpha_2) = \tfrac{1}{2}(N_1 - N_2) \tag{8.7}$$

it can be verified easily that these realizations obey the algebra

$$[H, X^\pm] = \pm X^\pm \qquad [X^+, X^-] = 2H \tag{8.8}$$

as required.

Since the $(\mathcal{A}_j, \alpha_j^\dagger)$s, or $(\alpha_j, \mathcal{A}_j^\dagger)$s, obey the same algebra as the boson operator pairs $(b_j, b_j^\dagger)$ (except for the Hermiticity property) one can also extend the other boson realizations of classical Lie algebras to obtain realizations in non-commutative spaces defined by the relations (2.5). For example, if we let

$$\begin{aligned} &K_+ = \mathcal{A}_1^\dagger \mathcal{A}_2^\dagger \quad (\text{or} \quad \alpha_1^\dagger \alpha_2^\dagger) \qquad K_- = \alpha_2 \alpha_1 \quad (\text{or} \quad \mathcal{A}_2 \mathcal{A}_1) \\ &K_0 = \tfrac{1}{2}(N_1 + N_2 + 1) \end{aligned} \tag{8.9}$$

then (K_+, K_-, K_0) form the $su(1,1)$ algebra.

9. Concluding remarks

The physical signifiance of system of q-oscillators is still only a subject of speculation. Here, we shall note a few interesting aspects of a q-bosonic system obeying the relations (2.5). Let the Hamiltonian be

$$H = \sum_{j=1}^{n} \epsilon_j A_j^\dagger A_j \tag{9.1}$$

with the levels (j) being non-degenerate; in the limit $q \to 1$, H would correspond to an n-level system of non-interacting bosons. The ordering of the A_js may be assumed to follow the ordering of the ϵ_js, $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_n$. The energy spectrum is given by

$$H|n_1, n_2, \ldots, n_j, \ldots n_n\rangle = E(n_1, n_2, \ldots, n_j, \ldots, n_n)|n_1, n_2, \ldots, n_j, \ldots, n_n\rangle$$

$$E(n_1, n_2, \ldots, n_j, \ldots, n_n) = \sum_j \epsilon_j q^{2\sum_{i<j} n_i} \left(\frac{q^{2n_j} - 1}{q^2 - 1}\right) \tag{9.2}$$

$$n_1, n_2, \ldots, n_j, \ldots, n_n = 0, 1, 2, \ldots.$$

The striking feature of this system is the correlation throughout the structure. When the occupation number of the jth level increases by one the excitation energy

$$\begin{aligned} \Delta_j E(n_1, n_2, \ldots, n_j, \ldots, n_n) \\ &= E(n_1, n_2, \ldots, (n_j + 1), \ldots, n_n) - E(n_1, n_2, \ldots, n_j, \ldots, n_n) \\ &= \epsilon_j q^{2\sum_{i\leqslant j} n_i} + \sum_{l>j} \epsilon_l q^{2\sum_{k<l} n_k} (q^{2n_l} - 1) \end{aligned} \tag{9.3}$$

depends on the population of all the levels and the energy parameters (ϵ_l) of the levels higher than jth. In the limit $q \to 1, \Delta_j E = \epsilon_j$, independent of the population of the levels, as it should be for an assembly of non-interacting bosons. It is quite interesting to note that if $0 < q^2 < 1$ the system behaves as if there is an attractive collective interaction: more crowd $\to$ easier excitation. If $q^2 > 1$ the system seems to possess a repulsive collective interaction. In both cases $(q^2 - 1)$ indicates a measure of the strength of this interaction.

The average occupation number of the jth level will be

$$\langle n_j\rangle = \frac{\sum_{n_j=0}^{\infty} n_j \exp\{-\beta E(n_1, n_2, \ldots, n_j, \ldots, n_n)\}}{\sum_{n_j=0}^{\infty} \exp\{-\beta E(n_1, n_2, \ldots, n_j, \ldots, n_n)\}} \tag{9.4}$$

when the occupation numbers of the other levels are fixed to be $(n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_n)$; this represents a truly conditional mean density at the particular level j given that all the other levels are occupied in a certain pattern. It is obvious that the distribution (9.4) reduces to the usual boson distribution in the limit $q \to 1$. In the case of independent, or single-mode, bosonic q-oscillators the distribution function has been discussed in the literature [41].

Now, suppose the states of the system have degeneracies. We shall take the Hamiltonian to be

$$H = \sum_{j=1}^{n} \epsilon_j \left(\sum_{k=1}^{g_j} A_{jk}^{\dagger} A_{jk} \right) \tag{9.5}$$

where the ordering of the level indices (j) is according to $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_n$ and the ordering of the degeneracy indices (k) for each j is fixed in some fashion. In this case the energy spectrum is given by

$$H|n_{11}, n_{12}, \ldots, n_{ng_n}\rangle = E((n)_1, (n)_2, \ldots, (n)_n)|n_{11}, n_{12}, \ldots, n_{ng_n}\rangle$$
$$(n)_j = \sum_{k=1}^{g_j} n_{jk} \qquad E((n)_1, (n)_2, \ldots, (n)_n) = \sum_{j=1}^{n} \epsilon_j q^{2\sum_{i<j}(n)_i} \left(\frac{q^{2(n)_j} - 1}{q^2 - 1} \right). \tag{9.6}$$

It should be noted that the energies depend only on the total occupation numbers of the levels irrespective of the way in which the states are ordered within each degenerate level.

There can be an alternative interpretation of the above Hamiltonian H. Using the boson realization of the As given in section 3 we can write

$$H = \sum_{j=1}^{n} \epsilon_j q^{2\sum_{i<j}\sum_{k=1}^{g_i} b_{ik}^{\dagger} b_{ik}} \left(\frac{q^{2\sum_{k=1}^{g_j} b_{jk}^{\dagger} b_{jk}} - 1}{q^2 - 1} \right) \tag{9.7}$$

the number operators are now $(b_{jk}^{\dagger} b_{jk})$. This H with the spectrum as in (9.6), may be thought of as a (quantum group inspired) model Hamiltonian for a system of interacting ordinary bosons, the interaction being effectively taken into account by the nonlinear dependence on the $(b_{jk}^{\dagger} b_{jk})$s and the phenomenological parameter q.

The discussion can be extended to a system with bosonic and fermionic degrees of freedom obeying relations of the type (7.2). The observations made above for the bosonic system are seen to be valid for the q-deformed boson–fermion system with the fermionic degrees of freedom obeying the Pauli principle; the effect of q-deformation is manifested through the appearance of additional interaction energy.

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